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SHAPOVALOV DETERMINANT FOR LOOP SUPERALGEBRAS

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For the Kac–Moody superalgebra associated with the loop superalgebra with values in a finite-dimensional Lie superalgebra $\mathfrak{g}$, we show what its quadratic Casimir element is equal to if the Casimir element for $\mathfrak{g}$ is known (if $\mathfrak{g}$ has an even invariant supersymmetric bilinear form). The main tool is the Wick normal form of the even quadratic Casimir operator for the Kac–Moody superalgebra associated with $\mathfrak{g}$; this Wick normal form is independently interesting. If $\mathfrak{g}$ has an odd invariant supersymmetric bilinear form, then we compute the cubic Casimir element. In addition to the simple Lie superalgebras $\mathfrak{g} = \mathfrak{g}(A)$ with a Cartan matrix A for which the Shapovalov determinant was known, we consider the Poisson Lie superalgebra $\mathfrak{poi}(0|n)$ and the related Kac–Moody superalgebra.

Keywords: Lie superalgebra, Shapovalov determinant

1. Introduction

1.1. Motivations: Physics. The justifications for calculating the Shapovalov determinant and Casimir element (in the nonsuper case) are expounded very lucidly in detail in [1]. We note that in various applications, the simple algebra that initiated the study is often less interesting than certain of its "relatives" (its nontrivial central extension, the algebra of derivations, and the result of iterating these constructions). In what follows, having the simple object at the center of our attention, we also keep its relatives in view.

1.1.1. Stringy algebras. In a seminal paper [2], Belavin, Polyakov, and Zamolodchikov observed that the infinite number of generators of the conformal group in the two-dimensional case generate Ward identities for correlation functions, and these differential equations (Ward identities) completely specify the behavior of the correlation functions. The components of the stress–energy operator in conformal field theory, together with the central charge, form the Virasoro algebra, and this reduces studying the conformal theory to studying the (irreducible) highest-weight representations of the Virasoro algebra.

A large class of conformal theories, the so-called minimal models, was explicitly constructed in [3]. In studying them, complete descriptions of the (irreducible) unitarizable highest-weight representations of a real form of the complex Virasoro algebra is exceptionally important.

1.1.2. Current algebras. Bosonization of free fermions with spin and the internal symmetry group G provides an example of a nontrivial conformal theory based on the Kac–Moody algebra $\widehat{\mathfrak{g}^{(1)}}$.[1] The components of the stress–energy operator for these theories, constructed from quadratic forms of fermion

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[1] Its formal definition (see [4]) is a bit complicated: $\widehat{\mathfrak{g}^{(1)}}$ is a certain subalgebra of the Lie algebra of derivations of the nontrivial central extension $\widetilde{\mathfrak{g}^{(1)}}$ of the loop algebra $\mathfrak{g}^{(1)}$ of maps $S^1 \to \mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie}(G)$ is a simple finite-dimensional complex Lie algebra (see (18) below).

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current operators for the loop group $G^{(1)}$, satisfy the relations for elements of the Virasoro algebra with the central charge $C = \dim \mathfrak{g}/(k + c_\mathfrak{g})$, where k is the value of the central charge of $\widehat{\mathfrak{g}^{(1)}}$ and $c_\mathfrak{g}$ is the value of the (quadratic) Casimir operator of $\mathfrak{g}$ in the adjoint representation [5]. The Hamiltonian of the WZW model can also be constructed with quadratic forms of current operators and thus also represents a nontrivial conformal field theory. The differential equations for the multipoint correlation functions of the WZW primary fields are the Knizhnik–Zamolodchikov equations introduced in [6].

In all these cases, the irreducible representations are described using the Shapovalov determinant. We recall its definition and the ways to compute it below.

1.1.3. Super versions. The Shapovalov determinant has been computed for (relatives of) simple stringy superalgebras, but not for all cases (a recent paper [7] claims to solve all the cases, but it has omissions). Here, we consider the Kac–Moody superalgebras; the stringy superalgebras are considered in our next paper [8].

1.2. Ways to calculate the Shapovalov determinant: Mathematics. The Casimir elements. It is clear from the considerations above that the Casimir elements (elements of the center of the universal enveloping algebra or its completion) are very important. In many problems, it suffices to know only the quadratic elements, but we must have them explicitly.

The Shapovalov determinant is a useful tool for verifying the irreducibility of certain representations of Lie algebras and Lie superalgebras $\mathfrak{g}$ with a vacuum vector (with highest or lowest weight), namely, the Verma modules, and even for constructing certain irreducible modules with a vacuum vector.

To define the Shapovalov determinant, the algebra $\mathfrak{g}$ must be rather "symmetric," i.e.,

> $\mathfrak{g}$ must have a Cartan subalgebra (a maximal nilpotent subalgebra coinciding with its normalizer) $\mathfrak{t}$ whose even part is commutative and diagonalizes $\mathfrak{g}$ and is such that the weight-zero subspace of $\mathfrak{g}$ (with respect to $\mathfrak{t}_{\bar{0}}$) coincides with $\mathfrak{t}$; $\mathfrak{g}$ must have an involutive (or superinvolutive if we use the sign rule; see (2)) antiautomorphism σ that interchanges the root vectors (with respect to $\mathfrak{t}_{\bar{0}}$) of opposite sign. (1)

We recall that an *antiautomorphism* of a Lie superalgebra $\mathfrak{g}$ is a linear map $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ that for any $x, y \in \mathfrak{g}$, satisfies

$$\sigma\big([x,y]\big) = \begin{cases} (-1)^{p(x)p(y)}[\sigma(y), \sigma(x)] & \text{if we use the sign rule,} \\ [\sigma(y), \sigma(x)] & \text{if we ignore the sign rule.} \end{cases} \tag{2}$$

An endomorphism σ is said to be *involutive* if $\sigma^2 = \mathrm{Id}$ (where Id is the identity operator) and *superinvolutive* if $\sigma^2(x) = (-1)^{p(x)}x$. Because the Shapovalov determinant is defined up to a scalar factor and using or ignoring the sign rule only affects its sign, we can choose the more convenient definition in each case. In computing the Shapovalov determinant, it is usually more convenient to ignore the sign rule.

In addition, the finiteness condition should be satisfied:

> the root spaces of $\mathfrak{g}$ are finite-dimensional; the number of partitions of any weight in $U(\mathfrak{g}^+)$ (see (9)) into positive roots of $\mathfrak{g}$ is finite. (3)

Nothing beyond (1) and (3) is needed for determining the Shapovalov determinant, but it is much easier to compute it in the presence of the even quadratic Casimir element C_2 (of the center of $U(\mathfrak{g})$ or its completion[2] $\widehat{U}(\mathfrak{g})$) or (which is not quite the same if $\dim \mathfrak{g} = \infty$, but it suffices for our purposes) in the presence of a nondegenerate *even* bilinear form B on $\mathfrak{g}$. *In the presence of such an element C_2, the Shapovalov determinant is a product of linear terms* (see [9]–[12] for various cases where this statement is proved).

[2]The hat over U means that the elements of $\widehat{U}(\mathfrak{g})$ are possibly infinite sums of elements of $U(\mathfrak{g})$.

1.3. Cases where an even quadratic Casimir exists. Kac and Kazhdan [9] computed the Shapovalov determinant for any Lie algebra with a symmetrizable Cartan matrix (with arbitrary (complex) entries); their technique is literally applicable to Lie superalgebras with a symmetrizable Cartan matrix (this was shown in [10] and expressed in detail in [11]). Moreover, Kac and Kazhdan used the Shapovalov determinant to describe the Jantzen filtration (see, e.g., [13] for the Jantzen filtration) of Verma modules over the Lie algebras they considered.

The technique of Kac and Kazhdan can be applied even in the absence of a symmetrizable Cartan matrix; it only makes computing the needed values of the operator C_2 easier. It is therefore reasonable to look around for Lie (super)algebras without a symmetrizable Cartan matrix but with a nondegenerate even invariant bilinear form. Grozman and Leites considered all such examples of simple Lie superalgebras among $\mathbb{Z}$-graded superalgebras of polynomial growth and their relatives:[3] one stringy superalgebra $\mathfrak{k}^L(1|6)$ (physicists call it the $N{=}6$ Neveu–Schwarz superalgebra), finite-dimensional Poisson superalgebras $\mathfrak{poi}(0|2n)$, and Kac–Moody algebras associated with $\mathfrak{poi}(0|2n)$. More precisely, [10] contains the results of computing the quadratic Casimir elements C_2 and describes the irreducible Verma modules over $\mathfrak{k}^L(1|6)$ and $\mathfrak{poi}(0|2n)$.

The study started in [10] was continued in [13], where a more explicit expression of the Shapovalov determinant than in [10] was given. The substantial nontrivial result in [13] is an explicit description of the Jantzen filtration for the Verma modules over $\mathfrak{poi}(0|2n)$.

There are many (perhaps, indescribably many) examples of filtered Lie (super)algebras of polynomial growth (i.e., such that the associated graded Lie (super)algebras grow polynomially) that have a C_2 (see, e.g., [14], [15]). The Shapovalov determinant has been computed only for one (the simplest) of such algebras, namely, for $\mathfrak{gl}(\lambda)$, where $\lambda \in \mathbb{C}$, and only for the simplest types of Verma modules [12].

1.4. Cases where no even quadratic Casimir element exists.

1.4.1. There is a nondegenerate odd invariant bilinear form. The queer Lie superalgebra $\mathfrak{q}(n)$ (a nontrivial superanalogue of $\mathfrak{gl}(n)$), the Poisson superalgebra $\mathfrak{poi}(0|2n+1)$, and the Kac–Moody superalgebras associated with them have a nondegenerate odd invariant bilinear form. In the 1990s, Grozman and Leites conjectured that because $\mathfrak{q}(n)$ and $\mathfrak{poi}(0|2n+1)$ are superanalogues of $\mathfrak{gl}(n)$, it is possible to compute their Shapovalov determinant (which is even difficult to define in these and similar cases), but they erred in computing $\mathfrak{q}(n)$ and decided that the terms into which it factors can be of any degree. Gorelik skillfully used the features of the "anticenter" and published an elegant proof that the Shapovalov determinant for $\mathfrak{q}(n)$ factors into the product of linear polynomials [16].

The simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth that have a nondegenerate odd invariant bilinear form are the following: exactly one stringy Lie superalgebra (we consider it in [8]) and also $\mathfrak{q}(n)$, $\mathfrak{poi}(0|2n+1)$, and the Kac–Moody algebras associated with them.

1.4.2. There is no nondegenerate bilinear form. Most of the stringy Lie (super)algebras [17] and the Lie (super)algebras of the type $\mathfrak{q}(n)^{(2)}$ with nonsymmetrizable Cartan matrices, recently considered in [18], have no nondegenerate bilinear form.

1.5. Our results and open problems. There primarily remains to be considered the yet unconsidered Lie (super)algebras with properties (1) and (3). The Kac–Moody (super)algebras associated with the loop (super)algebras with values in "symmetric" Lie (super)algebras are examples of such (super)algebras, and we therefore explicitly describe the Wick normal form of the Casimir operator for the Kac–Moody (super)algebras $\widehat{\mathfrak{g}^{(1)}}$ (and for "twisted" Kac–Moody algebras $\widehat{\mathfrak{g}^{(r)}}$) in terms of the Casimir operator for a

[3]For the same reasons that Kac–Moody algebras are "better" than simple loop algebras, the finite-dimensional Poisson Lie superalgebras $\mathfrak{poi}(0|n)$ are "better" than their simple relatives $\mathfrak{h}'(0|n)$, where $\mathfrak{h}(0|n) = \mathfrak{poi}(0|n)/\mathfrak{center}$ and $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

finite-dimensional simple Lie (super)algebra $\mathfrak{g}$. This also implies a description of the Shapovalov determinant for $\widehat{\mathfrak{g}^{(1)}}$, which was previously known only for Kac–Moody (super)algebras with a Cartan matrix. We also conjecture the form of the Shapovalov determinant for the Kac–Moody (super)algebra associated with $\mathfrak{poi}(0|2n)$ for $n > 2$.

We show that if $\mathfrak{g}$ has a nondegenerate *odd* invariant symmetric bilinear form, then $U(\mathfrak{g})$ (or $\widehat{U(\mathfrak{g})}$) contains a *cubic* central element (if $\dim \mathfrak{g} < \infty$, then this, as a rule, implies that the Shapovalov determinant factors into a product of factors of degree not exceeding two). We also conjecture the form of the Shapovalov determinant for $\mathfrak{poi}(0|2n+1)$.

2. Background on Lie superalgebras

We refer to [19] for a detailed background. The ground field is $\mathbb{C}$ in what follows.

2.1. The Poisson superalgebra. Let $G(m)$ be the Grassmann superalgebra generated by $\theta_1, \dots, \theta_m$. The Poisson Lie superalgebra (not to be confused with the Poisson–Lie (super)algebra) $\mathfrak{poi}(0|m)$ has the same superspace as $G(m)$, and the (Poisson) bracket is given by

$$\{f, g\}_{\text{P.b.}} = (-1)^{p(f)} \sum_{j \le m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j} \quad \text{for any } f, g \in \mathbb{C}[\theta]. \tag{4}$$

It is often more convenient to redesignate the θ and set

$$\begin{aligned} &\xi_j = \frac{1}{\sqrt{2}}\big(\theta_j - \sqrt{-1}\,\theta_{r+j}\big), \qquad \eta_j = \frac{1}{\sqrt{2}}\big(\theta_j + \sqrt{-1}\,\theta_{r+j}\big) \quad \text{for } j \le r = \left[\frac{m}{2}\right], \\ &\theta = \theta_{2r+1} \quad \text{if } m = 2r+1 \end{aligned} \tag{5}$$

(this is possible over $\mathbb{C}$ but not over $\mathbb{R}$, and brackets (4) and (6) are therefore not equivalent over $\mathbb{R}$). Accordingly, we modify the bracket (if $m = 2r$, then there is no term with θ):

$$\{f, g\}_{\text{P.b.}} = (-1)^{p(f)} \bigg(\sum_{j \le r} \bigg(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} \bigg) + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \bigg). \tag{6}$$

The quotient of $\mathfrak{poi}(0|n)$ by the center is the Lie superalgebra $\mathfrak{h}(0|n)$ of Hamiltonian vector fields generated by the functions

$$H_f := (-1)^{p(f)} \bigg(\sum_{j \le r} \bigg(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial \xi_j} \bigg) + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \bigg). \tag{7}$$

2.2. The integral. The still used notation $d^n\theta := d\theta_1 \cdots d\theta_n$ for the volume element in the Berezin integral is totally wrong (as was already clear in 1966 from the explicit form of the Berezinian of the Jacobi matrix of the coordinate change). Moreover, the volume form in the super case is not a differential form but a particular case of an integral form. A reasonable notation (with compulsory indication of the coordinates) is $\mathrm{vol}(\theta)$ (the motivations in the general case are given in [19]).

On $\mathfrak{poi}(0|n)$, more precisely, on the superspace of generating functions, the integral (equal to the coefficient of the leading term in the Taylor series in θ) determines a nondegenerate invariant bilinear form of the same parity as n:

$$(f\,|\,g) := \int f g \,\mathrm{vol}(\theta).$$

2.3. Cartan subalgebras, maximal tori, roots, and coroots. It was shown in [20], [21] that the Cartan subalgebras of any finite-dimensional simple and certain nonsimple (such as $\mathfrak{poi}$, $\mathfrak{q}$, and their subquotients) Lie superalgebra are conjugate by inner automorphisms. We always fix a Cartan subalgebra $\mathfrak{t}$ (e.g., for $\mathfrak{poi}(0|2n)$, we take $\mathfrak{t} = \mathbb{C}[\xi_1\eta_1, \dots, \xi_n\eta_n]$). For any $\alpha \in \mathfrak{t}_{\bar 0}^*$, we let $\mathfrak{g}_\alpha$ denote the set of $x \in \mathfrak{g}$ such that $\big(\mathrm{ad}_h - \alpha(h)\big)^N(x) = 0$ for a sufficiently large $N \in \mathbb{N}$ and any $h \in \mathfrak{t}_{\bar 0}$. Then we can easily show that

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}_{\bar 0}^*} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{t} \subset \mathfrak{g}_0. \tag{8}$$

Remark 2.1. In what follows, we consider only the algebras for which $\mathfrak{t} = \mathfrak{g}_0$.

We let R denote the set of nonzero functionals $\alpha \in \mathfrak{t}_{\bar 0}^*$ for which $\dim \mathfrak{g}_\alpha \neq 0$; this set is called the system of *roots* of $\mathfrak{g}$. For the Lie (super)algebras we consider, there exists an $H \in \mathfrak{t}_{\bar 0}$ such that $\alpha(H) \in \mathbb{R} \setminus \{0\}$ for any $\alpha \in R$. This property allows splitting the roots into positive and negative roots by setting

$$R^+ := \{\alpha \in R \,|\, \alpha(H) > 0\}, \qquad R^- := \{\alpha \in R \,|\, \alpha(H) < 0\}, \qquad \mathfrak{g}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha. \tag{9}$$

If $\mathfrak{t} = \mathfrak{t}_{\bar 0}$ and $\mathfrak{t}$ is commutative, then we can identify $U(\mathfrak{t})$ with $S(\mathfrak{t})$. Let HC be the Harish-Chandra projection, i.e., the projection on the direct summand $U(\mathfrak{t})$ in the decomposition

$$U(\mathfrak{g}) \simeq U(\mathfrak{t}) \oplus \big(\mathfrak{g}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^+\big) \quad \to \quad U(\mathfrak{t}) \simeq S(\mathfrak{t}). \tag{10}$$

2.4. The Shapovalov determinant.

Case 1. If the algebra $\mathfrak{t}$ is purely even and commutative, then for a fixed $\lambda \in \mathfrak{t}^*$, we set $M^\lambda = U(\mathfrak{g})/I$, where I is the left ideal generated by $\mathfrak{g}^+$ and the elements $h - \lambda(h)$, where $h \in \mathfrak{t}$. The $\mathfrak{g}$-module M^λ is called the *Verma module* with the highest weight λ. Obviously, as spaces, $M^\lambda \simeq U(\mathfrak{g}^-)m^\lambda$, where m^λ is the vacuum vector; the $U(\mathfrak{g}^-)$-action on m^λ is faithful (the result of the action of any nonzero element is nonzero). Antiautomorphism (2) can be (uniquely) extended to an antiautomorphism of $U(\mathfrak{g})$ using the relation

$$\sigma(x_1 \otimes \dots \otimes x_k) = \begin{cases} (-1)^{\sum_{i<j} p(x_i)p(x_j)} \sigma(x_k) \otimes \dots \otimes \sigma(x_1) & \text{using the sign rule,} \\ \sigma(x_k) \otimes \dots \otimes \sigma(x_1) & \text{ignoring the sign rule.} \end{cases} \tag{11}$$

On $U(\mathfrak{g}^-)$, we define the bilinear form $(\cdot\,|\,\cdot)$ by setting

$$(X \,|\, Y) = \mathrm{HC}\big(\sigma(X)Y\big) \quad \forall X, Y \in U(\mathfrak{g}^-) \tag{12}$$

with values in $U(\mathfrak{t}) = S(\mathfrak{t})$. Obviously, different weight subspaces of $U(\mathfrak{g}^-)$ are mutually orthogonal with respect to this bilinear form. Each determinant Sh_χ of the Gram matrix of the restriction of the bilinear form $(\cdot\,|\,\cdot)$ to the subspace $U(\mathfrak{g}^-)(-\chi)$ of weight $-\chi$ is called the *Shapovalov determinant*. Because $S(\mathfrak{t})$ is a commutative algebra, these determinants are well defined (as polynomials in $\mathfrak{t}$); if we do not fix a basis of $U(\mathfrak{g}^-)(-\chi)$ (and we do not fix bases in what follows), then they are determined up to a scalar factor.

On M^λ, we define the bilinear form (with values in $\mathbb{C}$)

$$(Xm^\lambda \,|\, Ym^\lambda)_\lambda = (X \,|\, Y)(\lambda) \equiv \mathrm{HC}\big(\sigma(X)Y\big)(\lambda) \quad \text{for any } X, Y \in U(\mathfrak{g}^-). \tag{13}$$

This definition is natural in the sense that for any X, X', Y, and Y' in $U(\mathfrak{g})$ (not only in $U(\mathfrak{g}^-)$), if $Xm^\lambda = X'm^\lambda$ and $Ym^\lambda = Y'm^\lambda$, then

$$\mathrm{HC}\big(\sigma(X)Y\big)(\lambda) = \mathrm{HC}\big(\sigma(X')Y'\big)(\lambda).$$

Obviously,

1. elements with distinct weights of M^λ are mutually orthogonal with respect to $(\cdot|\cdot)_\lambda$, and

2. the restriction of the bilinear form $(\cdot|\cdot)_\lambda$ to the subspace $M^\lambda(\lambda-\chi)$ of elements of weight $\lambda-\chi$ is degenerate if and only if $\mathrm{Sh}_\chi(\lambda)=0$.

Statement 2.1. 1. *Every nontrivial submodule of M^λ contains a nonzero element v whose weight differs from λ and $\mathfrak{g}^+v=0$ (such an element v is called* a singular vector *of M^λ).*

2. *If v is a singular vector of M^λ, then $U(\mathfrak{g}^-)v$ is a nontrivial submodule of M^λ.*

3. *An element $x\in M^\lambda$ is an isotropic element of the form $(\cdot|\cdot)_\lambda$ if and only if x can be represented as Av, where $A\in U(\mathfrak{g}^-)$ and v is a singular vector of M^λ.*

4. *If $v\in M^\lambda$ is a singular vector of weight μ and C is a central element of $U(\mathfrak{g})$, then* $\mathrm{HC}(C)(\lambda)=\mathrm{HC}(C)(\mu)$.

By virtue of this statement, describing all irreducible Verma modules is equivalent to computing all Shapovalov determinants. Moreover, we see that Casimir elements help to compute Shapovalov determinants.

Case 2. The case where $\mathfrak{t}_{\bar 1}\neq 0$ is considered in more detail in Sec. 5.1. Among finite-dimensional simple Lie superalgebras, only $\mathfrak{psq}(n)$ and $\mathfrak{h}'(0|2n+1)$ have this property. Gorelik [16] considered the case of $\mathfrak{psq}(n)$ and its relatives, and the case where $\mathfrak{h}'(0|2n+1)$ might theoretically be obtained from Gorelik's results by contraction, but we need an explicit formula and not a discussion. Below, we formulate a conjecture on the form of the Shapovalov determinant for $\mathfrak{poi}(0|2n+1)$; we conjecture that the answer for $\mathfrak{h}'(0|2n+1)$ for $n>1$ is analogous.

3. Casimir operators on Lie superalgebras: The case of an even invariant form

In this section, $\mathfrak{g}$ is a finite-dimensional Lie superalgebra, and $(\cdot|\cdot)$ is a nondegenerate even invariant supersymmetric bilinear form on $\mathfrak{g}$. Let $\{e_i\}_{i=1}^d$ be a basis of $\mathfrak{g}$. We set $a_{ij}=(e_i|e_j)$ and let $(b_{ij})=(a_{ij})^{-1}$ be the inverse matrix.

Statement 3.1. *The quadratic element*

$$\Omega_0=\sum_{i,j}b_{ij}e_i\otimes e_j\in U(\mathfrak{g}) \tag{14}$$

is a central element of $U(\mathfrak{g})$, i.e.,

$$[x,\Omega_0]=0\quad \textit{for all } x\in\mathfrak{g}. \tag{15}$$

Proof. It suffices to prove (15) for any $x\in\{e_i\}_{i=1}^d$. Let c_{ij}^k be the structure constants in this basis. Let $p_i=p(e_i)$ be the parities of the basis elements. We have

$$\begin{aligned}[e_k,\Omega_0]&=\sum_{i,j}b_{ij}\big([e_k,e_i]\otimes e_j+(-1)^{p_ip_k}e_i\otimes[e_k,e_j]\big)=\\&=\sum_{i,j}b_{ij}\sum_l\big(c_{ki}^le_l\otimes e_j+(-1)^{p_ip_k}c_{kj}^le_i\otimes e_l\big)=\\&=\sum_{i,j}\sum_l\big(b_{lj}c_{kl}^i+(-1)^{p_ip_k}b_{il}c_{kl}^j\big)e_i\otimes e_j.\end{aligned} \tag{16}$$

Hence, to prove the statement, it suffices to show that

$$\sum_l \big(b_{lj}c^i_{kl} + (-1)^{p_ip_k} b_{il}c^j_{kl}\big) = 0 \quad \text{for all } i,j,k \in \{1,2,\dots,d\}. \tag{17}$$

The invariance of the form $(\,\cdot\,|\,\cdot\,)$ implies that

$$0 = ([e_p,e_k]\,|\,e_r) - (e_p\,|\,[e_k,e_r]) = \sum_s (c^s_{pk}a_{sr} - c^s_{kr}a_{ps}) \quad \text{for all } k,p,r \in \{1,2,\dots,d\}.$$

We multiply this equality by $b_{ip}b_{rj}$ and sum over p and r. Because $(a_{ij})(b_{ij}) = 1_d$, we obtain

$$0 = \sum_p c^j_{pk}b_{ip} - \sum_r c^i_{kr}b_{rj} = \sum_l (c^j_{lk}b_{il} - c^i_{kl}b_{lj}) = -\sum_l \big((-1)^{p_kp_l}c^j_{kl}b_{il} + c^i_{kl}b_{lj}\big).$$

Because $(\,\cdot\,|\,\cdot\,)$ is even, the matrix (b_{ij}) is also even, and the first term in the last sum can be nonzero only if $p_i = p_l$. Therefore, the last equality is equivalent to condition (17). $\square$

For a given loop superalgebra $\mathfrak{g}^{(1)} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, we construct the *Kac–Moody Lie superalgebra* $\widehat{\mathfrak{g}^{(1)}} = \mathrm{Span}(\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}],u,z)$, where $u = t\,d/dt$ and z are even and the bracket is given by

$$\begin{aligned}
&[t^mx,t^ny] = t^{m+n}[x,y] + m\delta_{m,-n}(x\,|\,y)z \quad \text{for all } x,y\in\mathfrak{g}, \quad m,n\in\mathbb{Z},\\
&[z,X] = 0 \quad \text{for all } X \in \widehat{\mathfrak{g}^{(1)}},\\
&[u,t^nx] = nt^nx \quad \text{for all } x\in\mathfrak{g}, \quad n\in\mathbb{Z}.
\end{aligned} \tag{18}$$

We set

$$\Omega' = \sum_{n=-\infty}^{\infty}\sum_{i,j} b_{ij}t^ne_i\otimes t^{-n}e_j \in \widehat{U}\big(\widehat{\mathfrak{g}^{(1)}}\big). \tag{19}$$

Statement 3.2. *The bracket $[X,2u\otimes z+\Omega']$ is equal to zero for any $X\in\widehat{\mathfrak{g}^{(1)}}$.*

Proof. This equality is easily verified for $X = u,z$. Now let $X = t^me_k$. We compute $[X,\Omega']$ up to terms with z and obtain

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\sum_{i,j} b_{ij}\big(t^{m+n}[e_k,e_i]\otimes t^{-n}e_j + (-1)^{p_ip_k}t^ne_i\otimes t^{m-n}[e_k,e_j]\big) =\\
&\qquad = \sum_{n=-\infty}^{\infty}\sum_{i,j} b_{ij}\big(t^{m+n}[e_k,e_i]\otimes t^{-n}e_j + (-1)^{p_ip_k}t^{m+n}e_i\otimes t^{-n}[e_k,e_j]\big) =\\
&\qquad = \sum_{n=-\infty}^{\infty} \text{“}(t^{m+n}\otimes t^{-n})\cdot[e_k,\Omega_0]\text{”} = 0,
\end{aligned} \tag{20}$$

where the term in quotation marks is understood as follows: having represented $[e_k,\Omega_0]$ as the sum of quadratic elements, in each of them, we multiply the first factor by t^{m+n} and the second by t^{-n}.

We now compute the terms with z in $[X,\Omega']$. We can obtain these terms only from the terms in the sum with $n = \pm m$, and their sum is therefore equal to

$$\begin{aligned}
&\sum_{i,j} b_{ij}\big(m(e_k\,|\,e_i)z\otimes t^me_j + (-1)^{p_kp_i}mt^me_i\otimes(e_k\,|\,e_j)z\big) =\\
&\qquad = m\sum_{i,j} b_{ij}\big(a_{ki}z\otimes t^me_j + (-1)^{p_k(p_i+p_j)}t^me_i\otimes a_{kj}z\big).
\end{aligned} \tag{21}$$

Because $b_{ij} \neq 0$ only for $p_i = p_j$ and because $(a_{ij})(b_{ij}) = 1_d$, expression (21) becomes

$$m\biggl(\sum_j \delta_{kj} z \otimes t^m e_j + \sum_i \delta_{ik} t^m e_i \otimes z\biggr) = m(z \otimes X + X \otimes z) = 2mX \otimes z.$$

Because $[X, 2u \otimes z] = -2mX \otimes z$, we obtain $[X, 2u \otimes z + \Omega'] = 0$. $\square$

We represent the quadratic central element of $\widehat{U}\bigl(\widehat{\mathfrak{g}^{(1)}}\bigr)$ in the *Wick normal form*, i.e., such that the first factor in every tensor product has a nonpositive degree with respect to u and the second factor has a nonnegative degree. The Wick normal form of an operator Ω is denoted by $:\!\Omega\!:$. We set

$$\Omega^{pm} = \sum_{n=1}^{\infty} \sum_{i,j} b_{ij} t^{-n} e_i \otimes t^n e_j.$$

We compute $[X, \Omega_0 + 2\Omega^{pm} + 2u \otimes z]$. This expression obviously vanishes for $X = u$ or $X = z$; therefore, we set $X = t^m x$. A computation similar to the preceding one shows that $[X, 2u \otimes z]$ cancels with monomials in $[X, \Omega_0 + 2\Omega^{pm}]$ containing z, and we hence need only compute $[X, \Omega_0 + 2\Omega^{pm}]$ up to elements with z. If $X \in \mathfrak{g}$, then X commutes with Ω_0 (as previously shown) and, similarly, with each term in the sum over n in Ω^{pm}. We consider the case $X = t^m x$, where $m > 0$. From $[X, \Omega^{pm}]$, we obtain (see (20))

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{i,j} b_{ij} \bigl(t^{m-n}[e_k, e_i] \otimes t^n e_j + (-1)^{p_i p_k} t^{-n} e_i \otimes t^{m+n}[e_k, e_j]\bigr) = \\
&\qquad = \sum_{n=1}^{\infty} \sum_{i,j} b_{ij} t^{m-n}[e_k, e_i] \otimes t^n e_j + \sum_{n=m+1}^{\infty} \sum_{i,j} (-1)^{p_i p_k} t^{m-n} e_i \otimes t^n [e_k, e_j] = \\
&\qquad = \sum_{n=1}^{m} \sum_{i,j} b_{ij} t^{m-n}[e_k, e_i] \otimes t^n e_j + \sum_{n=m+1}^{\infty} \text{“}(t^{m+n} \otimes t^{-n}) \cdot [e_k, \Omega_0]\text{”} = \\
&\qquad = \sum_{n=1}^{m} \sum_{i,j} b_{ij} t^{m-n}[e_k, e_i] \otimes t^n e_j.
\end{aligned}$$

From $[X, \Omega_0 + \Omega^{pm}]$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{i,j} b_{ij} \bigl(t^{m-n}[e_k, e_i] \otimes t^n e_j + (-1)^{p_i p_k} t^{-n} e_i \otimes t^{m+n}[e_k, e_j]\bigr) = \\
&\qquad = \sum_{n=0}^{\infty} \sum_{i,j} b_{ij} t^{m-n}[e_k, e_i] \otimes t^n e_j + \sum_{n=m}^{\infty} \sum_{i,j} (-1)^{p_i p_k} t^{m-n} e_i \otimes t^n [e_k, e_j] = \\
&\qquad = -\sum_{n=0}^{m-1} \sum_{i,j} b_{ij} (-1)^{p_i p_k} t^{m-n} e_i \otimes t^n [e_k, e_j] + \sum_{n=m}^{\infty} \text{“}(t^{m+n} \otimes t^{-n}) \cdot [e_k, \Omega_0]\text{”} = \\
&\qquad = -\sum_{n=0}^{m-1} \sum_{i,j} b_{ij} (-1)^{p_i p_k} t^{m-n} e_i \otimes t^n [e_k, e_j].
\end{aligned}$$

Changing the summation indices $n \mapsto m - n$ and $i \leftrightarrow j$ in the last expression, we obtain

$$-\sum_{n=1}^{m} \sum_{i,j} b_{ji} (-1)^{p_j p_k} t^n e_j \otimes t^{m-n}[e_k, e_i] = -\sum_{n=1}^{m} \sum_{i,j} b_{ij} (-1)^{p_j (p_i + p_k)} t^n e_j \otimes t^{m-n}[e_k, e_i].$$

Hence, for $m > 0$, we have

$$[t^m e_k, \Omega_0 + 2\Omega^{pm} + 2u \otimes z] =$$

$$= \sum_{n=1}^{m} \sum_{i,j} b_{ij} \big(t^{m-n}[e_k, e_i] \otimes t^n e_j - (-1)^{p_j(p_i+p_k)} t^n e_j \otimes t^{m-n}[e_k, e_i] \big) =$$

$$= \sum_{n=1}^{m} \sum_{i,j} t^m b_{ij} \big[[e_k, e_i], e_j\big] = m t^m \sum_{i,j} b_{ij} \big[[e_k, e_i], e_j\big].$$

We obtain the similar result for $m < 0$. We hence have the following assertion.

Statement 3.3. *If the map*

$$A\colon x \mapsto \sum_{i,j} b_{ij} \big[[x, e_i], e_j\big] \tag{22}$$

is equal to $\lambda\,\mathrm{Id}$ *on* $\mathfrak{g}$*, where* $\lambda \in \mathbb{C}$*, then*

1. *the element*

$$\Omega = \Omega_0 + 2\Omega^{pm} + 2u \otimes z + \lambda u$$

   is central in $\widehat{U}\big(\widehat{\mathfrak{g}^{(1)}}\big)$,

2. *both* Ω_0 *and* Ω *can be represented in the Wick normal form, and*

3. *the linear terms of* $:\!\Omega_0\!:$ *and* $:\!\Omega\!:$ *differ by* λu.

Remark 3.1. Point 2 in Statement 3.3 holds because Ω_0 is a finite sum and $2\Omega^{pm} + 2u \otimes z + \lambda u$ is already in the normal form.

Conjecture. *If A is not scalar, then no nonzero central quadratic element of* $\widehat{U}\big(\widehat{\mathfrak{g}^{(1)}}\big)$ (*if there is such*) *can be expressed in the Wick normal form.*

Statement 3.4. *The map A commutes with the $\mathfrak{g}$-action.*

Proof. According to (17), we have

$$[e_k, Ax] = \sum_{i,j} b_{ij} \Big(\big[\big[[e_k, x], e_i\big], e_j\big] + (-1)^{p_k p(x)} \big[\big[x, [e_k, e_i]\big], e_j\big] +$$

$$+ (-1)^{p_k \left(p(x)+p_i\right)} \big[[x, e_i], [e_k, e_j]\big] \Big) =$$

$$= A[e_k, x] + (-1)^{p_k p(x)} \sum_{i,j} b_{ij} \big(\big[\big[x, [e_k, e_i]\big], e_j\big] + (-1)^{p_k p_i} \big[[x, e_i], [e_k, e_j]\big] \big)$$

and

$$\sum_{i,j} b_{ij} \big(\big[\big[x, [e_k, e_i]\big], e_j\big] + (-1)^{p_k p_i} \big[[x, e_i], [e_k, e_j]\big] \big) =$$

$$= \sum_{i,j,l} b_{ij} \big(c_{ki}^{l} \big[[x, e_l], e_j\big] + (-1)^{p_k p_i} c_{kj}^{l} \big[[x, e_i], e_l\big] \big) =$$

$$= \sum_{i,j} \Big(\sum_{l} \big(b_{lj} c_{kl}^{i} + (-1)^{p_k p_i} b_{il} c_{kl}^{j} \big) \Big) \big[[x, e_i], e_j\big] = 0.$$

We see that $[e_k, Ax] = A[e_k, x]$ for all $k \in \{1, 2, \ldots, d\}$ and $x \in \mathfrak{g}$. $\square$

Remark 3.2. From Schur's lemma, we deduce that if $\mathfrak{g}$ is a simple Lie (super)algebra, then A is a scalar operator. For example, if $\mathfrak{g} = \mathfrak{sl}(m|n)$ and $(X|Y) = \operatorname{tr} XY$, then $A = 2(m-n)$. We note that if $\mathfrak{g}$ is a direct sum of simple (super)algebras, then A is a direct sum of the corresponding scalar operators, and it hence might not be a scalar.

Statement 3.5. *Let $\mathfrak{g} = \mathfrak{poi}(0|2n)$. If $n > 1$, then $A = 0$; if $n = 1$, then A is not a scalar operator.*

Proof. If $n = 1$, then direct computation shows that the action of A is not zero on homogeneous elements of degrees one and two, while $A\mathbf{1} = 0$ (because the element $\mathbf{1} \in \mathfrak{poi}(0|2n)$ is central).

We note that if x and y are homogenous polynomials such that $(x|y) \neq 0$, then $\deg x + \deg y = 2n$. Therefore, for any homogenous polynomial X, if $AX \neq 0$, then $\deg AX = \deg X + 2n - 4$.

If $n = 2$, then $A(\theta_1\theta_2\theta_3\theta_4)$ can only be the zero element. Indeed, this element must be either zero or a homogenous polynomial of degree four, but the only such polynomial (up to a scalar factor), $\theta_1\theta_2\theta_3\theta_4$, is not in $\mathfrak{g}'$. Any element of the basis can be represented in the form

$$\frac{\partial}{\partial\theta_{i_1}} \cdots \frac{\partial}{\partial\theta_{i_k}}\theta_1\theta_2\theta_3\theta_4 = \bigl\{\theta_{i_1}, \bigl\{\ldots, \{\theta_{i_k}, \theta_1\theta_2\theta_3\theta_4\}\bigr\}\ldots\bigr\}, \quad k = 0, 1, 2, 3, 4,$$

where $\{\cdot\,,\cdot\}$ is the Poisson bracket, $k = 0$ means no differentiation, and A commutes with the $\mathfrak{poi}(0|2n)$-action. We hence conclude that $A = 0$.

If $n \geq 3$, then $AX = 0$ for any homogenous polynomial X of a degree not less than five. Because by bracketing with a polynomial of degree five, we can obtain any polynomial of a degree not exceeding four and because A commutes with the algebra action, we deduce that $A = 0$. $\square$

3.1. Twisted Kac–Moody (super)algebras. We now consider the case where the Lie (super)algebra $\mathfrak{g}$ as a linear space can be represented as $\tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$ with

$$[\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_0] + [\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{g}}_1] \subset \tilde{\mathfrak{g}}_0, \qquad [\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_1] \subset \tilde{\mathfrak{g}}_1, \qquad (\tilde{\mathfrak{g}}_0|\tilde{\mathfrak{g}}_1) = 0.$$

In other words, the tilde indicates a $\mathbb{Z}/2$-grading on $\mathfrak{g}$ (a priori having nothing in common with parity) respected by the (nondegenerate and even) invariant bilinear form $(\cdot|\cdot)$. In this case, we can define a twisted Kac–Moody superalgebra

$$\widehat{\mathfrak{g}^{(2)}} = \operatorname{Span}(\tilde{\mathfrak{g}}_0 \otimes \mathbb{C}[t^2, t^{-2}], \tilde{\mathfrak{g}}_1 \otimes t\mathbb{C}[t^2, t^{-2}], u, z),$$

where $u = t\,d/dt$ and z are even, with bracket (18) (see [22] for a list of simple twisted Kac–Moody algebras and [23] for a list of simple twisted Kac–Moody superalgebras).

Let A be a scalar operator $\lambda\,\mathrm{Id}$, as before. Let $\{e_i^0\}_{i=1}^{d_0}$ be a basis of $\tilde{\mathfrak{g}}_0$ and $\{e_i^1\}_{i=1}^{d_1}$ be a basis of $\tilde{\mathfrak{g}}_1$. We set $a_{ij}^0 = (e_i^0|e_j^0)$, $a_{ij}^1 = (e_i^1|e_j^1)$ and $(b_{ij}^0) = (a_{ij}^0)^{-1}$, $(b_{ij}^1) = (a_{ij}^1)^{-1}$. We introduce the notation

$$\begin{aligned}
&\Omega_0' = \sum_{i,j} b_{ij}^0 e_i^0 \otimes e_j^0 \in U(\tilde{\mathfrak{g}}_0),\\
&\Omega^{pm} = \sum_{n=1}^{\infty}\biggl(\sum_{i,j} b_{ij}^0 t^{-2n} e_i^0 \otimes t^{2n} e_j^0 + \sum_{i,j} b_{ij}^1 t^{-2n+1} e_i^1 \otimes t^{2n-1} e_j^1\biggr) \in \widehat{U}\bigl(\widehat{\mathfrak{g}^{(2)}}\bigr),\\
&\Omega = \Omega_0' + 2\Omega^{pm} + 2u \otimes z + \lambda u.
\end{aligned} \tag{23}$$

We note that $\tilde{\mathfrak{g}}_0$ is a subalgebra of $\mathfrak{g}$ and Ω_0' can be computed for $\tilde{\mathfrak{g}}_0$ the same as Ω_0 was computed for $\mathfrak{g}$. Then, as in the preceding section, we can prove the following assertion.

Statement 3.6. *The element* Ω *(see (23)) is a central element in* $\widehat{U}\big(\widehat{\mathfrak{g}^{(2)}}\big)$*; its linear part* $:\!\Omega\!:$ *is equal to the sum of the linear parts of* $:\!\Omega_0'\!:$ *and* λu*, i.e.,*

$$\text{the linear part of } :\!\Omega\!: \text{ (not counting } \lambda u\text{) depends only on } \tilde{\mathfrak{g}}_0. \tag{24}$$

Similarly, if the algebra $\mathfrak{g}$ has a $\mathbb{Z}/r$-grading $\mathfrak{g} = \bigoplus_{s=0}^{r-1} \tilde{\mathfrak{g}}_s$, then we can construct the algebra

$$\widehat{\mathfrak{g}^{(r)}} = \operatorname{Span}\left(\bigoplus_{s=0}^{r-1} \tilde{\mathfrak{g}}_s \otimes t^s \mathbb{C}[t^r, t^{-r}], u, z \right)$$

with bracket (18). As before, we introduce the following notation: e_i^s are the basis elements in $\tilde{\mathfrak{g}}_s$, $a_{ij}^s = (e_i^s \,|\, e_j^s)$, $(b_{ij}^s) = (a_{ij}^s)^{-1}$, and

$$\Omega_0' = \sum_{i,j} b_{ij}^0 e_i^0 \otimes e_j^0 \in U(\tilde{\mathfrak{g}}_0);$$

$$\Omega^{pm} = \sum_{s=0}^{r-1} \sum_{n=1}^{\infty} \sum_{i,j} b_{ij}^s t^{-2n+s} e_i^s \otimes t^{2n-s} e_j^s \in \widehat{U}\big(\widehat{\mathfrak{g}^{(r)}}\big).$$

If A is a scalar operator $\lambda \operatorname{Id}$, then $\Omega = \Omega_0' + 2\Omega^{pm} + 2u \otimes z + \lambda u$ is a central element.

But for the simple finite-dimensional algebras having such a grading with $r > 2$ (e.g., $\mathfrak{sl}(2m+1|2n+1)$ with the automorphism "minus supertransposition,"[4] $\mathfrak{g} = \mathfrak{osp}\big(4|2; \sqrt[3]{1}\big)$, and $\mathfrak{o}(8)$ with the grading induced by the third-order automorphism), *this grading is incompatible with the weight ones*. Therefore, the Cartan subalgebra of the Lie (super)algebra $\widehat{\mathfrak{g}^{(r)}}$ must be constructed not from the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ but from the Cartan subalgebra of $\tilde{\mathfrak{g}}_0$, which may have nothing in common with $\mathfrak{h}$. For example, if $\mathfrak{g} = \mathfrak{sl}(2m+1|2n+1)$, then $\tilde{\mathfrak{g}}_0 \simeq \mathfrak{o}(2m+1) \oplus \mathfrak{o}(2n+1)$ and $\mathfrak{h} \cap \tilde{\mathfrak{g}}_0 = \{0\}$.

Therefore, although the Casimir elements of $\mathfrak{g}$ and $\widehat{\mathfrak{g}^{(r)}}$ are similar, their Shapovalov determinants have nothing in common!

4. Casimir operators on Lie superalgebras: The case of an odd invariant form

In this section, $\mathfrak{g}$ is a finite-dimensional Lie superalgebra with a nondegenerate *odd* invariant supersymmetric (this is the same as just symmetric in this case) bilinear form $(\cdot\,|\,\cdot)$. As before, let $\{e_i\}_{i=1}^d$ be a basis of $\mathfrak{g}$, $a_{ij} = (e_i \,|\, e_j)$, and $(b_{ij}) = (a_{ij})^{-1}$ be the inverse matrix.

Statement 4.1. *The cubic element*

$$C_3 = \sum (-1)^{p_l} c_{ij}^k b_{im} b_{jl} e_k \otimes e_l \otimes e_m$$

is central in $U(\mathfrak{g})$*.*

The proof is similar to that of Statement 3.1.

[4]The order of this automorphism seems to be equal to four, but for all superdimensions except $(2m+1\,|\,2n+1)$, it is equal to two modulo the group of inner automorphisms (see [24], where all outer automorphisms of all finite-dimensional Lie superalgebras are listed).

Remark 4.1. Although the element C_3 seems less symmetric than the element Ω in Statement 3.1, we can show that the coefficient

$$F(k,l,m)=\sum(-1)^{p_l}c_{ij}^k b_{im}b_{jl}$$

of $e_k\otimes e_l\otimes e_m$ obeys the sign rule applied to $e_k\otimes e_l\otimes e_m$ with respect to permutation of the indices k, l and m, i.e.,

$$F(k,l,m)=(-1)^{p_kp_l}F(l,k,m)=(-1)^{p_lp_m}F(k,m,l).$$

Therefore, if $\mathfrak{g}$ is not commutative, then the degree of C_3 in $U(\mathfrak{g})$ is exactly three (and not less).

For an algebra $\mathfrak{g}$ with an odd invariant form, we can construct the algebra $\widehat{\mathfrak{g}^{(1)}}=\mathrm{Span}(\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}],u,z)$ with relations (18) but with an *odd* z.

Conjecture. *If $\mathfrak{g}$ is a simple Lie superalgebra, then $\widehat{U}\big(\widehat{\mathfrak{g}^{(1)}}\big)$ has a central element of degree three that can be represented in the Wick normal form.*

5. Kac–Moody-type superalgebras based on queer Lie superalgebras

5.1. What to do if the Cartan subalgebra has odd elements. In this section, we consider the Shapovalov determinant for Lie superalgebras $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{t}$ such that $0<n_{\bar 1}:=\dim\mathfrak{t}_{\bar 1}<\infty$ and $\mathfrak{t}_{\bar 0}$ is commutative. Among $\mathbb{Z}$-graded simple Lie superalgebras of polynomial growth (and their relatives), the algebras with these properties are only the Ramond superalgebras $R(N)=\mathfrak{k}^M(1|N)$ [8], the relatives of the queer superalgebra $\mathfrak{q}(n)$ considered in [16], and also the algebras whose Shapovalov determinant nobody has yet considered: relatives of the Lie superalgebra $\mathfrak{poi}(0|2n+1)$ and the Kac–Moody algebras associated with both $\mathfrak{q}(n)$ and $\mathfrak{poi}(0|2n+1)$.

The Shapovalov form $(\,\cdot\,|\,\cdot\,)$ defined in (12) takes values in $U(\mathfrak{t})$. If $\mathfrak{t}$ is purely even, then $U(\mathfrak{t})=S(\mathfrak{t})$ is a commutative algebra, and the determinant of $(\,\cdot\,|\,\cdot\,)$ is well defined up to a scalar factor. If $\mathfrak{t}_{\bar 1}\neq 0$, then it is not even clear how to define the Shapovalov determinant. The simplest way to treat this problem if $\mathfrak{t}_{\bar 0}$ is commutative is to

$$\text{consider only Verma modules with a one-dimensional vacuum on which } \mathfrak{t}_{\bar 1} \text{ acts trivially.} \tag{25}$$

But this restriction significantly limits the possible weights of the vacuum vector: $[\mathfrak{t}_{\bar 1},\mathfrak{t}_{\bar 1}]$ then also acts trivially on the vacuum. If $[\mathfrak{t}_{\bar 1},\mathfrak{t}_{\bar 1}]=\mathfrak{t}_{\bar 0}$, then the *only* one-dimensional $\mathfrak{t}$-module up to parity is the trivial module.

It was stated in [16] that J. Bernstein suggested defining the Shapovalov determinant as follows. First, we note that $U(\mathfrak{t})$ is a Clifford algebra over $S(\mathfrak{t}_{\bar 0})$. For a one-dimensional $\mathfrak{t}_{\bar 0}$-module $\mathbb{C}(\lambda)$ of weight $\lambda\in\mathfrak{t}_{\bar 0}^*$, we define the *vacuum module* as

$$U(\mathfrak{t})\bigotimes_{U(\mathfrak{t}_{\bar 0})}\mathbb{C}(\lambda). \tag{26}$$

Clearly, $\mathrm{sdim}\,U(\mathfrak{t})\bigotimes_{U(\mathfrak{t}_{\bar 0})}\mathbb{C}(\lambda)=2^{n_{\bar 1}-1}\,|\,2^{n_{\bar 1}-1}$.

By the Poincaré–Birkhoff–Witt theorem, $U(\mathfrak{t})$ is a filtered algebra, and the associated graded algebra is $S(\mathfrak{t}_{\bar 0})\otimes\wedge(\mathfrak{t}_{\bar 1})$. Therefore, there is a natural map (the composition of the contraction $\mathfrak{q}\to\mathfrak{poi}$ with the Berezin integral)

$$\int : U(\mathfrak{t})\longrightarrow S(\mathfrak{t}_{\bar 0})\otimes\bigwedge^{\max}(\mathfrak{t}_{\bar 1})\cong S(\mathfrak{t}_{\bar 0}).$$

This map is defined up to a scalar factor, but this suffices for us. Hence, if $\mathfrak{t}_{\bar 1}\neq 0$, then we can give another form, *Bernstein's Shapovalov form*, by setting

$$B(\,\cdot\,|\,\cdot\,)=\int(\,\cdot\,|\,\cdot\,). \tag{27}$$

This form takes values in the commutative algebra $S(\mathfrak{t}_{\bar 0})$, and its determinant is therefore well-defined.

We consider the case where $n_{\bar 1} = 1$ separately. In this case, $U(\mathfrak{t})$ is a commutative (not supercommutative!) superalgebra. Therefore, if we use $U(\mathfrak{t}) \bigotimes_{U(\mathfrak{t}_{\bar 0})} \mathbb{C}(\lambda)$ for the vacuum, then the usual Shapovalov form has a determinant well defined up to a scalar factor. But this determinant is equal to the one defined and computed in [16] up to some power of a nonzero element of $\mathfrak{t}_{\bar 1}$.

5.2. An open problem. The form of the Shapovalov determinant depends on the system of simple roots; therefore, it is vital to know how to pass from one system to another. For finite-dimensional Lie algebras (and Kac–Moody algebras), the passage is achieved using elements of the (affine) Weyl group. For Lie superalgebras, such a passage is achieved using the "odd" reflections introduced first by Skornyakov and independently by Penkov and Serganova [21]. An interesting open problem is to describe systems of positive roots (or at least an algorithm for passing from one system to another and an explicit form of at least one system) for $\mathfrak{g} = \mathfrak{poi}$ or $\mathfrak{h}'$ (perhaps it is better to take the superalgebra $\mathfrak{g} = \mathfrak{poi}$ or $\mathfrak{h}'$ augmented by the grading operator).

6. Explicit formulas

6.1. Lie superalgebras with a symmetrizable Cartan matrix. Let $\mathfrak{g} = \mathfrak{g}(A, I)$ be a Lie superalgebra with a normalized symmetrizable Cartan matrix A and $\alpha_1, \dots, \alpha_n$ be the corresponding system of simple roots (then I is the sequence of parities of the root spaces corresponding to the roots $\alpha_1, \dots, \alpha_n$). The symmetrizability of A means that there exists an invertible diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ such that $B = DA$ is symmetric.

On the root lattice $\Delta = \operatorname{Span}_{\mathbb{Z}}(\alpha_1, \dots, \alpha_n)$, we define the following objects:

1. a symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ using the conditions

$$(\alpha_i, \alpha_j) = B_{ij} \quad \text{for all } i, j = 1, \dots, n,$$

2. a linear function[5] F using the conditions

$$F(\alpha_i) = \frac{1}{2} B_{ii} \quad \text{for any } i = 1, \dots, n,$$

and

3. the element

$$h_\gamma = \sum k_i d_i h_i \quad \text{for any} \quad \gamma = \sum k_i \alpha_i \in \Delta$$

(we note that generally $h_{\alpha_i} \neq h_i$).

Let $R \subset \Delta$ be a root system and R^+ be the system of positive roots. Because each root space is either purely even or purely odd, each root can be endowed with a parity. Let

$$\overline{R}^+_{\bar 0} = \{\alpha \in R^+ \,|\, p(\alpha) = \bar 0,\ \alpha/2 \notin R\}, \qquad \overline{R}^+_{\bar 1} = \{\alpha \in R^+ \,|\, p(\alpha) = \bar 1,\ 2\alpha \notin R\}.$$

[5]Essentially, the function F is a $\dim \mathfrak{g}{=}\infty$ analogue of the function $(\rho, \cdot)$, where (the summations take the multiplicities into account)

$$\rho := \frac{1}{2}\Big(\sum_{\alpha \in R^+_{\bar 0}} \alpha - \sum_{\alpha \in R^+_{\bar 1}} \alpha\Big),$$

where $R^+_{\bar 0}$ and $R^+_{\bar 1}$ are the respective subsets of R^+ consisting of even and odd roots, which means that the corresponding root vectors are even and odd.

We define the *partition function* (which was sometimes called the *Kostant function* when Kostant was actively working and is denoted by K in his honor) on the set of weights of $U(\mathfrak{g}^-)$ by setting

$$K(\mu) = \dim U(\mathfrak{g}^-)(\mu).$$

Theorem 6.1 [25], [11]**.** *For the $\mathfrak{g}(A,I)$-module M^χ, we have*

$$\mathrm{Sh}_\chi = \prod_{\alpha\in\overline{R}_{\bar 0}^+}\prod_{m\in\mathbb{N}}\Big(h_\alpha + F(\alpha) - \frac{m}{2}(\alpha,\alpha)\Big)^{K(\chi-m\alpha)} \times$$

$$\times \prod_{\alpha\in\overline{R}_{\bar 1}^+}\prod_{m=2k+1\in\mathbb{N}}\Big(h_\alpha + F(\alpha) - \frac{m}{2}(\alpha,\alpha)\Big)^{K(\chi-m\alpha)}. \tag{28}$$

Remark 6.1. This expression should not be used for χ that are not weights of $U(\mathfrak{g}^+)$ (for example, for $\chi = k\alpha$, where α is an odd simple root such that 2α is not a root and $k > 1$); in these cases, the formula may produce a wrong (nonscalar) result.

6.2. The algebra $\mathfrak{po}(0|2n+1)$. We let the variables be ξ_i, η_i, and θ, where $1 \le i \le n$, and consider the Poisson bracket of the form

$$[f,g]_{\mathrm{P.b.}} = (-1)^{p(f)}\sum_{i=1}^{n}\Big(\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial g}{\partial \xi_i} + \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta}\Big).$$

We choose the Cartan subalgebra whose generating functions are the elements of $\mathbb{C}[\xi_1\eta_1,\dots,\xi_n\eta_n,\theta]$. Let ε_i be the weight of ξ_i (and then the weight of η_i is $-\varepsilon_i$), where $i = 1,\dots,n$. For the weight $\gamma = \sum c_i\varepsilon_i$, we define

$$h_\gamma := \sum c_i\xi_1\eta_1\cdots\widehat{\xi_i\eta_i}\cdots\xi_n\eta_n.$$

We also introduce the element $h_{\max} := \xi_1\eta_1\cdots\xi_n\eta_n$.

6.2.1. Conjecture. *Let $\mathfrak{g} = \mathfrak{po}(0|2n+1)$, where $n > 2$. We choose one of the systems of positive roots containing all the ε_i. Then Sh_χ factors into the product of linear factors of the form h_γ (where the γ are positive roots of $\mathfrak{g}$ such that $(\chi-\gamma)$ are positive or zero weights) and $h_{\max}$.*

For $\mathfrak{poi}(0|3)$ and $\mathfrak{poi}(0|5)$, all systems of positive roots are conjugate, and it hence suffices to describe Sh_χ for one system. We choose the system consisting of ε_1 for $\mathfrak{poi}(0|3)$ and the system consisting of $\varepsilon_1, \varepsilon_2, \varepsilon_1 \pm \varepsilon_2$ for $\mathfrak{poi}(0|5)$.

We introduce the notation

$h_1 = \xi_1\eta_1$ and $h_0 = \mathbf{1}$ (the central element of $\mathfrak{po}(0|3)$) for $\mathfrak{poi}(0|3)$ and

$h_{10} = \xi_1\eta_1$, $h_{01} = \xi_2\eta_2$, and $h_{11} = \xi_1\eta_1\xi_2\eta_2$ for $\mathfrak{poi}(0|5)$.

6.2.2. Conjecture. *If $\mathfrak{g} = \mathfrak{poi}(0|3)$, then $\mathrm{Sh}_{k\varepsilon_1}$ factors into linear factors of the form h_0 and $h_1 - m/2$, where $m = 1,\dots,k$.*

6.2.3. Conjecture. *If $\mathfrak{g} = \mathfrak{poi}(0|5)$, then for $\chi = k\varepsilon_1 + l\varepsilon_2$ (such that χ is positive if $k, k+l \ge 0$ and at least one of the numbers k or l is positive), Sh_χ factors into linear factors of the form*

$$h_{11},$$

$$h_{01} \quad \text{if } k, k+l \ge 1,$$

$$h_{10} \quad \text{if } k+l \ge 1,$$

$$h_{01} - h_{10} + m \quad \text{for } m \in \mathbb{Z}, \quad 1 \le m \le k,$$

$$h_{01} + h_{10} - m \quad \text{for } m \in \mathbb{Z}, \quad 1 \le m \le k, \quad m \le \frac{k+l}{2}.$$

Remark 6.2. In Conjectures 6.2.2 and 6.2.3, the scalar summands are understood to be the scalar terms of $U\big(\mathfrak{poi}(0|2n+1)\big)$ and not the scalar (central) elements of $\mathfrak{poi}(0|2n+1)$ itself.

6.3. The algebra $\widehat{\mathfrak{po}(0|2n)}^{(1)}$. We let the variables be ξ_i and η_i, where $1 \le i \le n$, and take the bracket in $\mathfrak{po}(0|2n)$ of the form

$$[f,g]_{\text{P.b.}} = (-1)^{p(f)} \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i} \right).$$

We choose the Cartan subalgebra whose generating functions are the elements of $\mathbb{C}[\xi_1\eta_1, \ldots, \xi_n\eta_n] \oplus \operatorname{Span}(u,z)$. Let ε_i be the weight of ξ_i (then the weight of η_i is e_i), where $i = 1, \ldots, n$, and let ε' be the weight of t. For the weight $\gamma = \sum c_i\varepsilon_i + c'\varepsilon'$, we define

$$h_\gamma = \sum c_i \xi_1\eta_1 \cdots \widehat{\xi_i\eta_i} \cdots \xi_n\eta_n + c'z.$$

We also introduce the element $h_{\max} := \xi_1\eta_1 \cdots \xi_n\eta_n$.

6.3.1. Conjecture. *Let $\mathfrak{g} = \widehat{\mathfrak{po}(0|2n)}^{(1)}$, where $n > 2$, and let one of the system of positive roots containing all roots ε_i and ε' be chosen. Then Sh_χ factors into the product of linear factors of the form $h_{\max}$ and h_γ, where the γ are positive roots of $\mathfrak{g}$ such that $\chi - \gamma$ are either positive or zero weights.*

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